Interim Report No. 3
Contract No. NAS8-5411

PARAMETER OPTIMIZATION

July 1, 1964

UNIVERSITY OF ALABAMA RESEARCH INSTITUTE
Huntsville, Alabama

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"Parameter Optimization"

to

Director
George C. Marshall Space Flight Center
Huntsville, Alabama
Attn: M-P & C-MEA

by

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INTRODUCTION

The present report, which is the third of a series of interim reports, deals with the studies on "Parameter Optimization" carried out during the period, March 1, 1964, to June 30, 1964, under the contract NAS8-5411. It is a preliminary report and the results should not be considered final.

In the preceding report the effects of measurement errors of the observables on the orbital parameters in general tracking operations were discussed and the corresponding equations derived. Two groups of relationships were derived, namely, those where the equations contain only observables and those of another group where the orbital parameters were contained in the equations. The combinations of quantities involved in these relationships are instructively displayed in Tables 1 and 2 of Interim Report No. 2, pp. 11 and 12.

The error study is continued in this report and the results presented in Part I. A table similar to that of the preceding report shows again the combinations of quantities involved in the derived equations. The two sections of the table are shown in Part IV as Appendix. Part I contains the error study of the geometric case where only distances and angles as observed quantities are involved. In the present study, the equations contain only orbital parameters. The equations are considerably simpler than those containing only observables and they are better accessible for the evaluation. Section B and C deal with the dynamic case. As a supplement to the equations of Interim Report No. 2, the present relations contain only and in a few cases primarily measured quantities. The equations become more complicated but the relationships will have to be known for the later optimization of tracking operations.

In Section D errors are being considered applying normalized distances as one of the measured quantities. The distance is normalized with regard to the major axis of the elliptic orbit which is proportional to the total energy content. Since the normalization reduces the variable orbital parameters to one, namely the eccentricity, the results become very instructive. The derivations lead to relationships which clearly indicate regions and time periods along the orbit where the tracking errors will become small and where they will become excessive.

Part II deals with an attempt to minimize the errors of the orbital parameters caused by deviations of the observed quantities for the geometric case. The observed quantities are three positions along the orbit. The minimization of the error of the parameter p leads to conditions which are typical for a circular orbit. This orbit then represents a minimum-error orbit under the assumed conditions. Simultaneously with this analytic optimization a tentative computer study was begun as briefly described in Section B of Part II.

Heretofore the study was dealing with observables related to a geocentric coordinate system. Since the practical tracking operations will involve measurements to and from stations on the surface of the earth, coordinate transformations have to be taken into account. These coordinate transformations will on one hand directly affect the error relationships and the parameter errors, on the other hand, errors of the position of the ground stations will cause additional errors of the orbit determination. A review of the equations by which these transformations can be carried out is presented in Part III. A station-centered system, a vehicle born inertial, and a vehicle-born, orbit-aligned coordinate system are being discussed and the corresponding equations described in the three sections of this part.

NOTATION SHEET

e	Eccentricity
E	Total energy
L	Angular momentum
a	Major axis
θ_o	Orientation of major axis
G	Gravitational constant
M_e	Mass of earth
M_s	Mass of satellite
K	GM_e
C	$r^2 \dot{\theta}$ = Areal velocity = KP
ξ	r/a normalized radial distance
$\hat{e}_i$	Unit vectors
$\vec{\Omega}$	Rotation vector of earth
i	Inclination of orbital plane
Ω	Location of ascending node

I. CONTINUATION OF ERROR ANALYSIS

The orbital parameters of a given satellite orbit may be determined by making a number of measurements of the position, velocity, velocity components, etc., of the orbiting vehicle. In Report No. 1 the equations for determining an orbit from a minimum number of such measurements were given. Instrumental errors cause all the measurements to be uncertain to some extent. In the work of Report No. 2 some of the effects of Instrumental errors on the uncertainty of the orbital parameters were investigated. It should be pointed out that the error analysis may be taken from two approaches. (1) The equations of the errors of the orbital parameters due to deviations of the measured quantities may be derived from the general equations of the coordinates in terms of the orbital parameters, as in No. 2 part II B - C. (2) The equations for the errors of the orbital parameters due to deviations of the measured quantities may be derived from the equations for the parameter in question in terms of the measured quantities only. The equations contain only the results of measurements, as in No. 2 part II A.

The first method (1) gives equations which are simpler in form than those of the second method, but the results are not as readily applicable to the error introduced by a particular set of measurements, where errors can occur in several variables concurrently. Thus, for completeness, the following includes the error analysis for position measurements from the equations of the orbital parameters, and the analysis for the dynamic cases with only measured quantities or observables as variables. This completes the error analysis of the cases considered.

In Section C, normalization is applied for simplifying the error equations based on position measurements. In the dynamic case, based on position and velocity measurements, no similar simplification can be readily obtained.

A. Error Analysis for Position Measurements From the Orbit Equation

The equation for the orbit has been written in the form:

$$r = \frac{p}{1 + e \cos(\theta - \theta_o)}$$

where $$P = a(1 - e^2) = L^2/GM_eM_s^2$$
$$= C^2/K \text{ and } C = r^2\,\dot{\theta}\,;\; K = GM_e\ .$$

Consider now an error Δr in r. Writing the polar equation in the form

$$C^2 = Kr\,(1 + e\cos\theta)$$

we get $$2C\frac{\partial C}{\partial r} = K(1 + e\cos\theta) + K\cos\theta\,\frac{\partial e}{\partial r}$$

where $$\frac{\partial C}{\partial r} = 2r\dot{\theta} = \frac{2C}{r}$$

Thus $$K\cos\theta\,\frac{\partial e}{\partial r} = \frac{4C^2}{r} - \frac{C^2}{r} = \frac{3KP}{r}$$

and $$\frac{\partial e}{\partial r} = \frac{3P}{r\cos\theta} = \frac{3a(1-e^2)}{r\cos\theta}\ . \tag{1}$$

For the error in P we write

$$P = \frac{C^2}{K}$$

and $$\frac{\partial P}{\partial r} = \frac{2C}{K}\,\frac{\partial C}{\partial r} = \frac{4C^2}{rK} = \frac{4P}{r} \tag{1a}$$

or $$\frac{\partial P}{\partial r} = 4(1 + e\cos\theta)\ . \tag{2}$$

For the error in a we use the equation

$$P = a(1-e^2)$$

and get

$$\frac{\partial P}{\partial r} = (1 - e^2)\frac{\partial a}{\partial r} - 2\,ae\frac{\partial e}{\partial r} \quad .$$

Therefore

$$(1-e^2)\frac{\partial a}{\partial r} = \frac{4P}{r} + \frac{6\,ae\,P}{r\cos\theta} = \frac{2P\,(2\cos\theta + 3ae)}{r\cos\theta}$$

or

$$\frac{\partial a}{\partial r} = \frac{2\,a\,(2\cos\theta + 3\,ae)}{r\cos\theta} \quad . \qquad (3)$$

Finally, for the error in θ_o we write:

$$P = r\;[1 + e\cos(\theta - \theta_o)]$$

Then

$$\frac{\partial P}{\partial r} = 1 + e\cos(\theta - \theta_o) + er\sin(\theta - \theta_o)\frac{\partial\theta_o}{\partial r} + r\cos(\theta - \theta_o)\frac{\partial e}{\partial r}$$

or

$$\frac{3P}{r} - 3P = er\sin(\theta - \theta_o)\;\frac{\partial\theta_o}{\partial r}$$

which gives

$$\frac{\partial\theta_o}{\partial r} = \frac{3P}{er\sin(\theta - \theta_o)}\;[\,1/r - 1\,] \quad . \qquad (4)$$

Now consider an error $\Delta\theta$ in the other variable θ. We write the polar equation in the form

$$a\,(1 - e^2) = r\,(1 + e\cos\theta)$$

then

$$-2\,ae\frac{\partial e}{\partial\theta} = r\cos\theta\,\frac{\partial e}{\partial\theta} - re\sin\theta$$

or

$$\frac{\partial e}{\partial \theta} = \frac{re \sin \theta}{r \cos \theta + 2\, ae} \quad . \tag{5}$$

Similarly we have

$$\frac{\partial P}{\partial \theta} = r \cos \theta \, \frac{\partial e}{\partial \theta} - re \sin \theta$$

$$= \frac{er^2 \cos \theta \sin \theta}{r \cos \theta + 2\, ae} - re \sin \theta \quad .$$

Thus

$$\frac{\partial P}{\partial \theta} = \frac{2\, aer \sin \theta}{r \cos \theta + 2\, ae} \quad . \tag{6}$$

Finally for the error in θ_o we write

$$P = r \,[\, 1 + e \cos (\theta - \theta_o) \,]$$

and get

$$\frac{\partial P}{\partial \theta} = r \cos (\theta - \theta_o) \frac{\partial e}{\partial \theta} - re \sin (\theta - \theta_o) \frac{\partial \theta_o}{\partial \theta} \quad .$$

Substituting for $\frac{\partial P}{\partial \theta}$ and $\frac{\partial e}{\partial \theta}$ gives

$$re \sin (\theta - \theta_o) \frac{\partial \theta_o}{\partial \theta} = \frac{er^2 \cos (\theta - \theta_o) \sin (\theta - \theta_o) + 2\, ae^2 r \sin (\theta - \theta_o)}{r \cos (\theta - \theta_o) + 2\, ae}$$

$$\frac{\partial \theta_o}{\partial \theta} = \frac{r \cos (\theta - \theta_o) + 2\, ae}{r \cos (\theta - \theta_o) + 2\, ae} = 1 \tag{7}$$

From the equations above, one can determine the effect on the orbital parameters, of a small deviation in either of the independent variables r or θ.

B. Error Analysis for the Dynamic Case With Observables as Parameters

(1) Determinations from Radial Velocity Measurements.

From the results of part II of Report No. 1, we have the following expressions for the orbital parameters in terms of two radial velocity measurements at two positions on the orbit.

$$P = \frac{L^2}{M_s^2 M_e G} = \frac{1}{M_e G}\left[\frac{2GM_e\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \dot{r}_2^2 - \dot{r}_1^2}{\frac{1}{r_1^2} - \frac{1}{r_2^2}}\right]$$

$$= \frac{1}{M_e G}\left[2GM_e\left(\frac{r_1 r_2}{r_1 + r_2}\right) + \left(\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\right)\left(\dot{r}_2^2 - \dot{r}_1^2\right)\right] \qquad (8)$$

$$E = -\frac{GM_e M_s}{2a} = 1/2\ M_s\left(\dot{r}_1^2 + \frac{L^2}{M_s^2 r^2}\right) - \frac{GM_e M_s}{r_1} =$$

$$= 1/2\ M_s\left\{\dot{r}_1^2 + 2GM_e\left(\frac{r_1 r_2}{r_1 + r_2}\right) + \left(\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\right)\left(\dot{r}_2^2 - \dot{r}_1^2\right)\right\} - \frac{GM_e M_s}{r_1} \qquad (9)$$

$$e = \sqrt{1 + \frac{2EL^2}{M_s^2 M_e^2 G^2}} \qquad (10)$$

θ_o can be determined from one of the two equations

$$\theta_o = \theta_1 - \cos^{-1}\left[\frac{1}{e}\left(\frac{p}{r_1} - 1\right)\right] \qquad (11)$$

or

$$\theta_o = \tan^{-1}\left[\frac{\gamma\cos\theta_2 - \cos\theta_1}{\sin\theta_1 - \gamma\sin\theta_2}\right], \quad \gamma = \frac{r_2}{r_1}\left(\frac{L^2 - r_1 M_s M_e G}{L^2 - r_2 M_s^2 M_e G}\right) . \qquad (12)$$

For the error analysis we need the partial derivatives

$$\frac{\partial p}{\partial r_1}, \ \frac{\partial e}{\partial r_1}, \frac{\partial \theta_o}{\partial r_1}, \ \frac{\partial p}{\partial \dot r_1}, \ \frac{\partial e}{\partial \dot r_1}, \ \frac{\partial \theta_o}{\partial \dot r_1}, \ \frac{\partial \theta_o}{\partial \theta} \ .$$

Starting with the parameter P we have

$$P = \frac{L^2}{M_s^2 M_e G}$$

thus

$$\frac{\partial P}{\partial r_1} = \frac{1}{M_s^2 M_e G} \quad \frac{\partial L^2}{\partial r_1}$$

$$= \frac{1}{M_s^2 M_e G} \cdot M_s^2 \left\{ 2M_e G \left(\frac{r_2}{r_2 + r_1} - \frac{r_1 r_2}{(r_2 + r_1)^2} \right) + \right.$$

$$\left. + (\dot r_2^2 - \dot r_1^2) \left(\frac{2 r_1 r_2^2}{(r_2^2 - r_1^2)} + \frac{2 r_1^3 r_2^3}{(r_2^2 - r_1^2)^2} \right) \right\}$$

$$= \frac{1}{M_e G} \left\{ 2M_e G \frac{r_2^2}{r_2^2 + r_1^2} + (\dot r_2^2 - \dot r_1^2) \frac{r_1 r_2^4}{(r_2^2 - r_1^2)^2} \right\} . \tag{13}$$

For later use we note that

$$\frac{\partial L^2}{\partial r_1} = \frac{M_s^2}{M_e G} \left\{ 2M_e G \left(\frac{r_2^2}{r_2^2 + r_1^2} \right) + \left(\dot r_2^2 - \dot r_1^2 \right) \frac{r_1 r_2^4}{(r_2^2 - r_1^2)^2} \right\} . \tag{14}$$

For an error in $\dot{r}$ we need

$$\frac{\partial P}{\partial \dot{r}_1} = \frac{1}{M_s^2 M_e G} \quad \frac{\partial L^2}{\partial \dot{r}_1} = \frac{-2}{M_e G} \quad \dot{r}_1 \left(\frac{r_1^2 \, r_2^2}{r_2^2 - r_1^2} \right) \tag{15}$$

For later use we note that

$$\frac{\partial L^2}{\partial \dot{r}_1} = \frac{2 \dot{r}_1}{M_s^2 M_e G} \left(\frac{r_1^2 \, r_2^2}{r_2^2 - r_1^2} \right) \tag{16}$$

We consider next the eccentricity e which is defined by

$$e = \sqrt{1 + \frac{2 \, EL^3}{M_s^2 \, M_e^2 G^2}}$$

Noting also that

$$E = {}^1/2 \; M_s \left(\dot{r}^{\,2} + \frac{L^2}{M_s^2 \, r^2} \right) - \frac{GM_e M_s}{r}$$

we get

$$2e \frac{\partial e}{\partial r_1} = \frac{2 \, E}{M_s^2 \, M_e^2 G^2} \quad \frac{\partial L^2}{\partial r_1} + \frac{2L^2}{M_s^2 M_e^2 \; G^2} \quad \frac{\partial E}{\partial r_1}$$

$$= \frac{2}{M_s^2 M_e^2 G^2} \left[\frac{M_s}{2} \; (\dot{r}_1^{\,2} + \frac{L^2}{M_s^2 r_1^2}) - \frac{GM_e M_s}{r_1} \right] \frac{\partial L^2}{\partial r_1}$$

$$+ \frac{2L^2}{M_s^2 M_e^2 G^2} \left[\frac{1}{2M_s} \left(\frac{1}{r_1^2} \; \frac{\partial L^2}{\partial r_1} - \frac{\partial L^2}{r_1^3} \right) + \frac{GM_e M_s}{r_1^2} \right] L^2$$

$$= \frac{2}{M_s^2 K^2}\left\{\frac{1}{2M_s}\left[\frac{K}{r_1^2} - \frac{2L^2}{r_1^3}\right]L^2 + \left[\frac{M_s}{2}\left(\dot{r}_1^2 + \frac{2L^2}{M_s^2 r_1^2}\right) - \frac{M_s K}{r_1}\right]\frac{\partial L^2}{\partial r_1}\right\} \tag{17}$$

where $\frac{\partial L^2}{\partial r_1}$ is given by equation (14) . For the error in e from an error in $\dot{r}$

we need $\frac{\partial e}{\partial r_1}$. Thus

$$2e\frac{\partial e}{\partial \dot{r}_1} = \frac{2}{M_s^2 M_e^2 G^2}\left\{\frac{M_s}{2}\left(2\dot{r}_1 + \frac{1}{M_s^2 r_1^2}\frac{\partial L^2}{\partial r_1}\right)L^2\right.$$

$$\left. + \left[\frac{M_s}{2}\left(\dot{r}_1^2 + \frac{L^2}{M_s^2 r_1^2}\right) - \frac{GM_e M_s}{r_1}\right]\frac{\partial L^2}{\partial \dot{r}_1}\right\}$$

$$= \frac{2}{M_s K^2}\left\{\dot{r}_1 L^2 + M_s\left[\frac{\dot{r}_1^2}{2} + \frac{L^2}{M_s^2 r_1^2} - \frac{K}{r_1}\right]\frac{\partial L^2}{\partial \dot{r}_1}\right\} \tag{18}$$

where $\frac{\partial L^2}{\partial \dot{r}_1}$ is given by equation (16) .

Finally, for the third parameter, θ_o we use equation (12) and get:

$$\frac{\partial \theta_o}{\partial r_1} = \frac{\partial}{\partial r_1}\tan^{-1}\left[\frac{\gamma\cos\theta_2 - \cos\theta_1}{\sin\theta_1 - \gamma\sin\theta_2}\right]$$

$$= \frac{1}{1 + \left[\frac{\gamma\cos\theta_2 - \cos\theta_1}{\sin\theta_1 - \gamma\sin\theta_2}\right]^2}\left[\frac{\cos\theta_2}{\sin\theta_1 - \gamma\sin\theta_2} + \frac{\gamma\cos\theta_2 - \cos\theta_1}{(\sin\theta_1 - \gamma\sin\theta_2)^2}\sin\theta_2\right]\frac{\partial\gamma}{\partial r_1}$$

$$= \frac{\sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1}{1 + \gamma^2 - 2\gamma(\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2)} \frac{\partial \gamma}{\partial r_1}$$

This expression may be simplified by use of the trigonometric identities for the sum and difference of two angles. Thus

$$\frac{\partial \theta_o}{\partial r_1} = \frac{\sin(\theta_1 - \theta_2)}{1 + \gamma^2 - 2\gamma \cos(\theta_1 - \theta_2)} \frac{\partial \gamma}{\partial r_1} \qquad (19)$$

From the expression for γ ,

$$\gamma = \frac{r_2}{r_1} \cdot \frac{L^2 - r_1 M_s^2 M_e G}{L^2 - r_2 M_s^2 M_e G}$$

we get

$$\frac{\partial \gamma}{\partial r_1} = -\frac{r_2}{r_1^2} \cdot \frac{L^2 - r_1 M_s^2 M_e G}{L^2 - r_2 M_s^2 M_e G} + \frac{r_2}{r_1} \frac{1}{L^2 - r_2 M_s^2 M_e G} \cdot$$

$$\left[\frac{\partial L^2}{\partial r_1} - M_s^2 M_e G\right] - \frac{r_2}{r_1}\left(\frac{L^2 - r_1 M_s^2 M_e G}{(L^2 - r_2 M_s^2 M_e G)^2}\right) \frac{\partial L^2}{\partial r_1}$$

$$= -\frac{r_2}{r_1^2} \frac{L^2}{L^2 - r_2 M_s^2 M_e G} + \frac{r_2}{r_1} M_s^2 M_e G (r_1 - r_2) \frac{\partial L^2}{\partial r_1} \qquad (20)$$

where $\frac{\partial L^2}{\partial r_1}$ is obtained from Equation (14) . Thus we have:

$$\frac{\partial \theta_o}{\partial r_1} = \frac{\sin(\theta_1 - \theta_2)}{1 + \gamma^2 - 2\gamma \cos(\theta_1 - \theta_2)} \left[\frac{r_2}{r_1^2} \frac{L^2}{L^2 - r_2 M_s^2 M_e G}\right.$$

$$+ \frac{r_2}{r_1} M_s^2 M_e G (r_1 - r_2) \frac{\partial L^2}{\partial r_1}\Bigg] \tag{21}$$

For errors in $\dot{r}$ we need

$$\frac{\partial \theta_o}{\partial \dot{r}_1} = \frac{\sin(\theta_1 - \theta_2)}{1 + \gamma^2 + 2\gamma\cos(\theta_1 - \theta_o)} \frac{\partial \gamma}{\partial \dot{r}_1} \tag{22}$$

where

$$\frac{\partial \gamma}{\partial \dot{r}_1} = \frac{r_2}{r_1} \frac{1}{L^2 - r_2 M_s^2 M_e G} - \frac{L^2 - r_1 M_s^2 M_e G}{(L^2 - r_2 M_s^2 M_e G)^2} \frac{\partial L^2}{\partial \dot{r}_1}$$

$$= \frac{r_2}{r_1} \left[\frac{M_s^2 M_e G (r_1 - r_2)}{(L^2 - r_2 M_s^2 M_e G)^2} \right] \frac{\partial L^2}{\partial \dot{r}_1} \tag{23}$$

and $\frac{\partial L^2}{\partial \dot{r}_1}$ is obtained from Equation (16). Finally for errors in the angle θ we need

$$\frac{\partial \theta_o}{\partial \theta_1} = \frac{\partial}{\partial \theta_1} \tan^{-1} \left[\frac{\gamma\cos\theta_2 - \cos\theta_1}{\sin\theta_1 - \gamma\sin\theta_2} \right]$$

$$= \frac{1}{1 + \left[\frac{\gamma\cos\theta_2 - \cos\theta_1}{\sin\theta_1 - \gamma\sin\theta_2} \right]^2} \left[\frac{\sin\theta_1}{\sin\theta_1 - \gamma\sin\theta_2} - \frac{(\gamma\cos\theta_2 - \cos\theta_1)}{(\sin\theta_1 - \gamma\sin\theta_2)^2}\cos\theta_1 \right]$$

$$= \frac{1 - \gamma\cos(\theta_1 - \theta_2)}{1 - 2\gamma\cos(\theta_1 - \theta_2) + \gamma^2} \tag{24}$$

From the above analysis one can determine the effect on the orbital parameters of an error in any one of the measured quantities in the detemination from radial velocity measurements. We will defer until later the consideration of special cases.

(2) Determination From Angular Velocity Measurement

Let us consider the results of part 3 of No. 1 where the orbital parameters were determined from the measurement of the angular velocity of the orbiting vehicle. (This quantity might be determined from position measurements at two times, t_1 and t_2 as mentioned in No. 2.)

Thus we have

$$P = \frac{(r_1^2 \dot{\theta}_1)^2}{M_e G} = \frac{r_1^4 \dot{\theta}_1^2}{M_e G} \tag{25}$$

$$\theta_o = \tan^{-1}\left(\frac{\gamma \cos \theta_2 - \cos \theta_1}{\sin \theta_1 - \gamma \sin \theta_2}\right) \tag{26}$$

and

$$e = \frac{P/_{r_1} - 1}{\cos(\theta_1 - \theta_o)} = \left(\frac{r_1^3 \dot{\theta}_1^2}{M_e G} - 1\right) \sec(\theta_1 - \theta_o) \tag{27}$$

For the error analysis we need

$$\frac{\partial P}{\partial \dot{r}_1}, \frac{\partial P}{\partial \dot{\theta}}, \frac{\partial \theta_o}{\partial r_1}, \frac{\partial \theta_o}{\partial \theta}, \frac{\partial \theta_o}{\partial \dot{\theta}}, \frac{\partial e}{\partial r_1}, \frac{\partial e}{\partial \theta} .$$

Starting with the equation for P we have:

$$\frac{\partial P}{\partial r_1} = \frac{4 r_1^3 \dot{\theta}_1^2}{M_e G} \tag{28}$$

and

$$\frac{\partial P}{\partial \dot{\theta}_1} = \frac{2 r_1^4 \dot{\theta}_1^2}{M_e G} . \tag{29}$$

We note that the expression for θ_o is exactly the same as those considered in Part A where radial velocities are measured at θ_1 and θ_2. Thus we have already calculated

$$\frac{\partial \theta_o}{\partial r_1} \quad \text{and} \quad \frac{\partial \theta_o}{\partial \theta} \quad \text{(equations (21) and (24))}$$

Thus we need only calculate $\dfrac{\partial \theta_o}{\partial \dot{\theta}}$ for the present case. This is trivial since we see immediately

$$\frac{\partial \theta_o}{\partial \dot{\theta}} = 0 \; . \tag{30}$$

Thus the effects of errors in θ , $\dot{\theta}$, and r on the determination of θ_o is complete.

Finally we find the errors in e from Equation (27) . Thus

$$\frac{\partial e}{\partial r_1} = \frac{3\, r_1^2\, \dot{\theta}_1^2}{M_e G} \sec\,(\theta_1 - \theta_o) \tag{31}$$

$$\frac{\partial e}{\partial \dot{\theta}} = \frac{2\, r_1^3\, \dot{\theta}_1}{M_e G} \sec\,(\theta_1 - \theta_o) \tag{32}$$

and

$$\frac{\partial e}{\partial \theta_1} = \left(\frac{r_1^3\, \dot{\theta}_1^2}{M_e G} - 1 \right) \frac{\sin\,(\theta - \theta_o)}{\cos^2\,(\theta - \theta_o)} \tag{33}$$

From these equations one may obtain the errors in the orbital parameters which result from instrumental errors in the determination of the orbit by angular velocity measurements.

D. Errors Derived from Normalized Orbital Parameters

The geometrical orbit equation for $\theta_o = 0$, is

$$r = \frac{p}{1 + e \cos \theta} \quad , \tag{34}$$

where

$$p = a\,(1 - e^2) \quad .$$

In the following, we are going to nomalize all our quantities by the major-axis "a" . Later it will be shown that this approach gives a considerable simplification.

Dividing Eq. (34) by a, we find

$$\frac{r}{a} = \frac{p/a}{1 + \cos \theta}$$

$$= \frac{1 - e^2}{1 + e \cos \theta} \tag{35}$$

Introducing a new parameter ξ , which is the nomalized parameter, such that

$$\xi = \frac{r}{a}$$

yields

$$(1 + e \cos \theta)\ \xi = 1 - e^2 \quad . \tag{36}$$

Taking the partial derivative of Eq. (36) with respect to ξ gives

$$1 + e \cos \theta + \xi \cos \theta \frac{\partial e}{\partial \xi} = -2e \frac{\partial e}{\partial \xi} \quad .$$

Thus

$$\frac{\partial e}{\partial \xi} = \frac{-1 - e \cos\theta}{\xi \cos\theta + 2e} . \tag{37}$$

From the normalized geometric orbit equation we have

$$1 + e\cos\theta + \xi\cos\theta\,\frac{\partial e}{\partial \xi} = \frac{1}{a}\,\frac{\partial p}{\partial \xi} . \tag{38}$$

Substituting Eq. (37) into Eq. (38) we find

$$\frac{\partial p}{\partial \xi} = \frac{2pe(1 + e\cos\theta)}{(1-e^2)(\xi\cos\theta + 2e)} . \tag{39}$$

Similarly, the following results can be obtained for the error of the eccentricity and parameter p from partial differentiation with respect to the angle θ.

$$\frac{\partial e}{\partial \theta} = \frac{e\,\xi\,\sin\theta}{\cos\theta + 2e} . \tag{40}$$

$$\frac{\partial p}{\partial \theta} = \frac{-2e^2 a\,\xi\,\sin\theta}{\xi\cos\theta + 2e} \tag{41}$$

We can eliminate ξ from above equations by substituting

$$\xi = \frac{1 - e^2}{1 + e\cos\theta} .$$

This gives,

$$\frac{1}{p}\,\frac{\partial p}{\partial \xi} = \frac{2e(1 + e\cos\theta)^2}{1 + 2e^2(\cos\theta - 1) + 2e(1 - e^2)(1 + \cos\theta)} , \tag{42}$$

$$\frac{1}{p}\frac{\partial p}{\partial \theta} = \frac{-2e^2 \sin\theta}{2e + \cos\theta\,(1-e^2)} \quad , \tag{43}$$

where only e and θ are contained in the equations. It should be noted that $\frac{\partial \ln p}{\partial \xi}$ and $\frac{\partial \ln p}{\partial \theta}$ can be substituted for the left-hand sides respectively. Since p normalized with regard to "a" is a function of e only, this case can be included in the error analysis of e , so that

$$\frac{\partial e}{\partial \xi} = - \frac{(1 + e\cos\theta)^2}{2e + \cos\theta\,(1+e^2)} \quad , \tag{44}$$

$$\frac{\partial e}{\partial \theta} = \frac{e\,(1-e^2)\sin\theta}{2e + \cos\theta\,(1+e^2)} \quad . \tag{45}$$

Since we know

$$\xi = \frac{r}{a} = \frac{1-e^2}{1+e\cos\theta} \quad ,$$

$$e^2 + \xi\cos\theta\, e + \xi - 1 = 0 \quad ,$$

$$e = \frac{-\xi\cos\theta \pm \sqrt{\xi^2\cos^2\theta - 4(\xi - 1)}}{2} \quad ,$$

$$= -\frac{\cos\theta}{2} \pm q \quad ,$$

where

$$q = \sqrt{\frac{\xi^2\cos^2\theta}{4} - \xi + 1} \quad .$$

It is obvious that q should have a real value for a true orbit. Therefore

$$\frac{\xi^2 \cos^2 \theta}{4} - \quad + 1 > 0 .$$

Further, for an elliptical orbit, we should have

$$0 < e < 1$$

then

$$q > \frac{\cos \theta}{2} .$$

From equations of $\frac{\partial e}{\partial \xi}$ and $\frac{\partial e}{\partial \theta}$ we can eliminate e and in terms of ξ . This will give us

$$\frac{\partial e}{\partial \theta} = \frac{(2q - \xi \cos \theta) \xi \sin \theta}{2q} \tag{46}$$

$$\frac{\partial e}{\partial \xi} = \frac{2 + (2q - \xi \cos \theta) \cos \theta}{4q} \tag{47}$$

where

$$q = \sqrt{\frac{\xi^2 \cos^2 \theta}{4} - \xi + 1}$$

From here we can see that in the region of $0 < \xi < 1$. This will guarantee us an elliptical orbit and finite errors everywhere (q always real and $0 < e < 1,$) . By increasing ξ then this nice situation no longer exists. An elliptical orbit and finite value of errors will exist only within a certain region. This shows it will depend on ξ .

In the following, we show where the breaking points are.

It is known

$$\frac{\partial e}{\partial \xi}, \quad \frac{\partial e}{\partial \theta} \rightarrow \infty \qquad \text{As } q \rightarrow 0.$$

$$\frac{\xi^2 \cos^2 \theta}{4} - \xi + 1 = 0.$$

$$\theta = \cos^{-1}\left[\pm \frac{2}{\xi} \sqrt{\xi - 1}\right]$$

It is obvious that for $\xi < 1$ no infinite slope exists. The breaking points only exist when $\xi > 1$.

The following three figures show graphically characteristic properties of the preceding error relationships. These figures are based on normalization with regard to the major axis "a" with $\xi = r/a$ as one of the observed quantities. By this normalization only the eccentricity remains as variable orbital parameter. The figure shows the relationship among the normalized observed distance ξ, the observed angle θ, and its corresponding errors. In Figure 1 the relationship between eccentricity and observed angle θ is shown where ξ is a parameter. From the figure we can see that at $\xi \leq 1.0$ the eccentricity varies as a function of θ smoothly. Under the condition $\xi > 1$, the curve becomes piecewise discontinuous. For instance at $\xi = 1.2$ where $32^\circ \leq \theta \leq 148^\circ$ "e" becomes imaginary and no real solution exists. The dots at the ends of the curves indicate in the figure the limits of the real solution. Figure 2 and Figure 3 show the errors of the eccentricity as a function of the observed angle θ using ξ as a parameter. Similarly as in Figure 1 for $\xi \leq 1$, smooth continuous curves exist. For $\xi > 1$, the curves become discontinuous and have the same invalid regions as in Figure 1. This means that no elliptic orbit can be found at the positions indicated by the parameters in these regions.

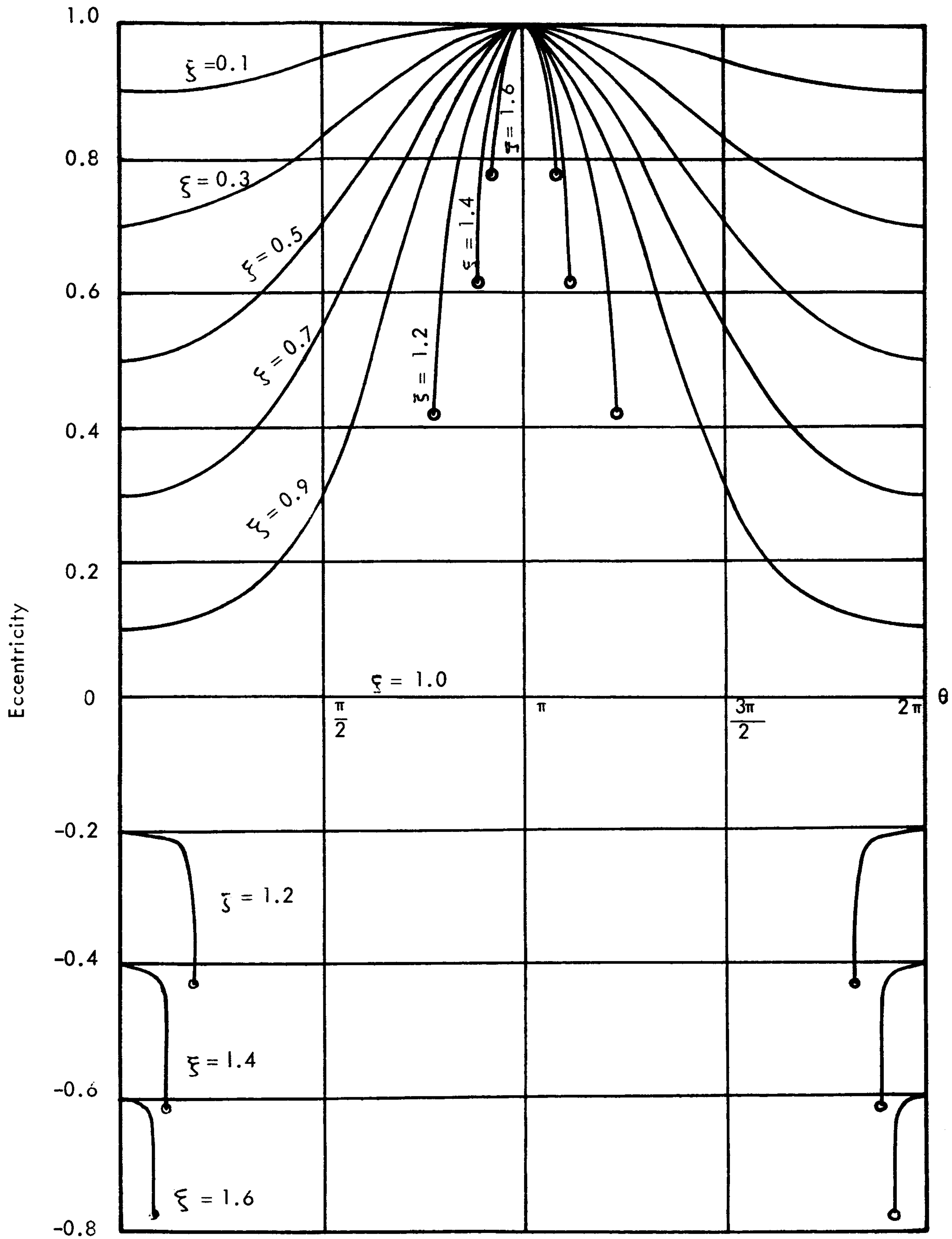


Figure 1 Values of e at which ξ = const. as a function of θ (notice at ξ = 1.2, 1.4, 1.6, we have a region which e is imaginary)

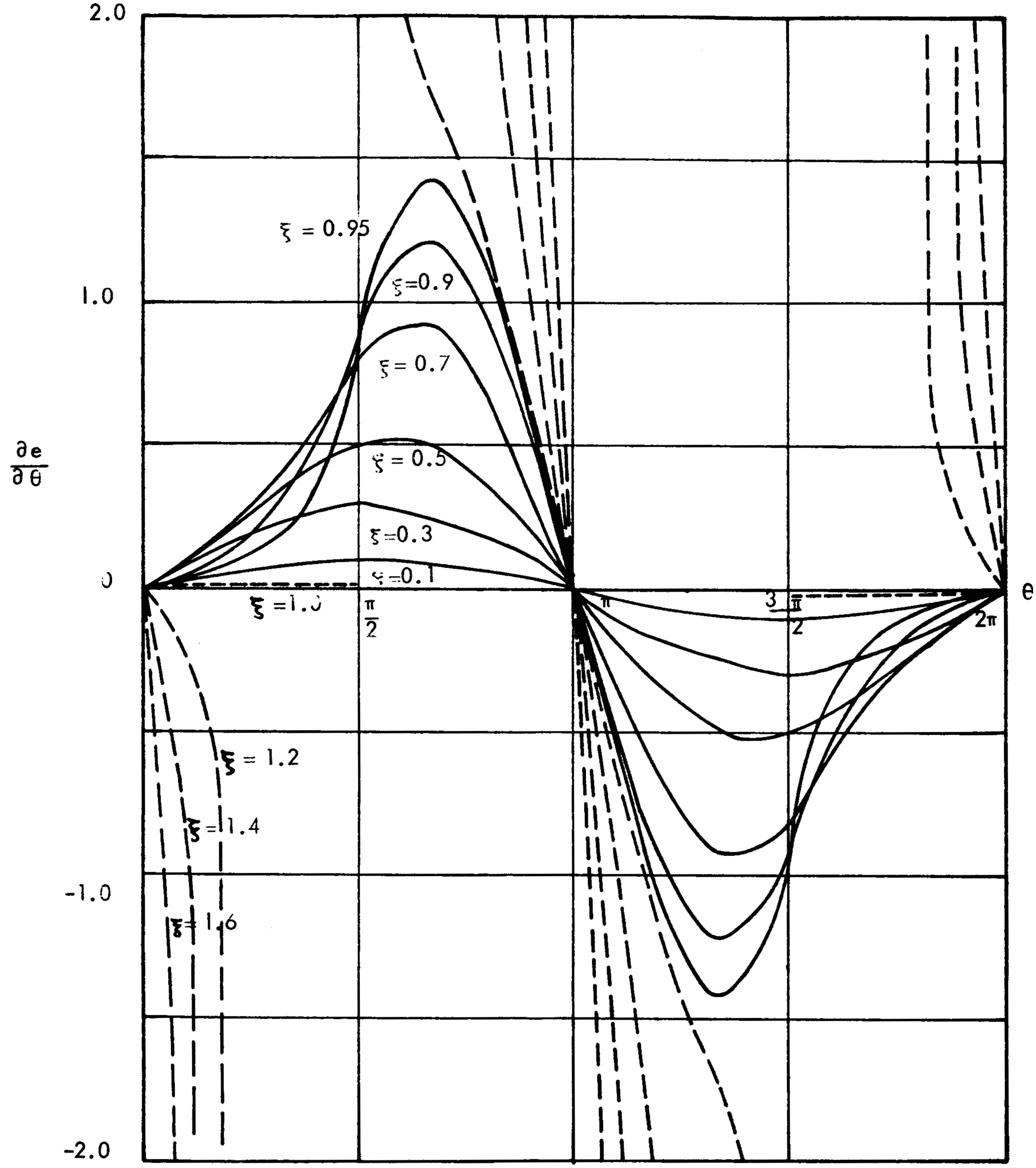


Figure 2 - Values of $\frac{\partial e}{\partial \theta}$ at which ξ = const. as a function of θ.

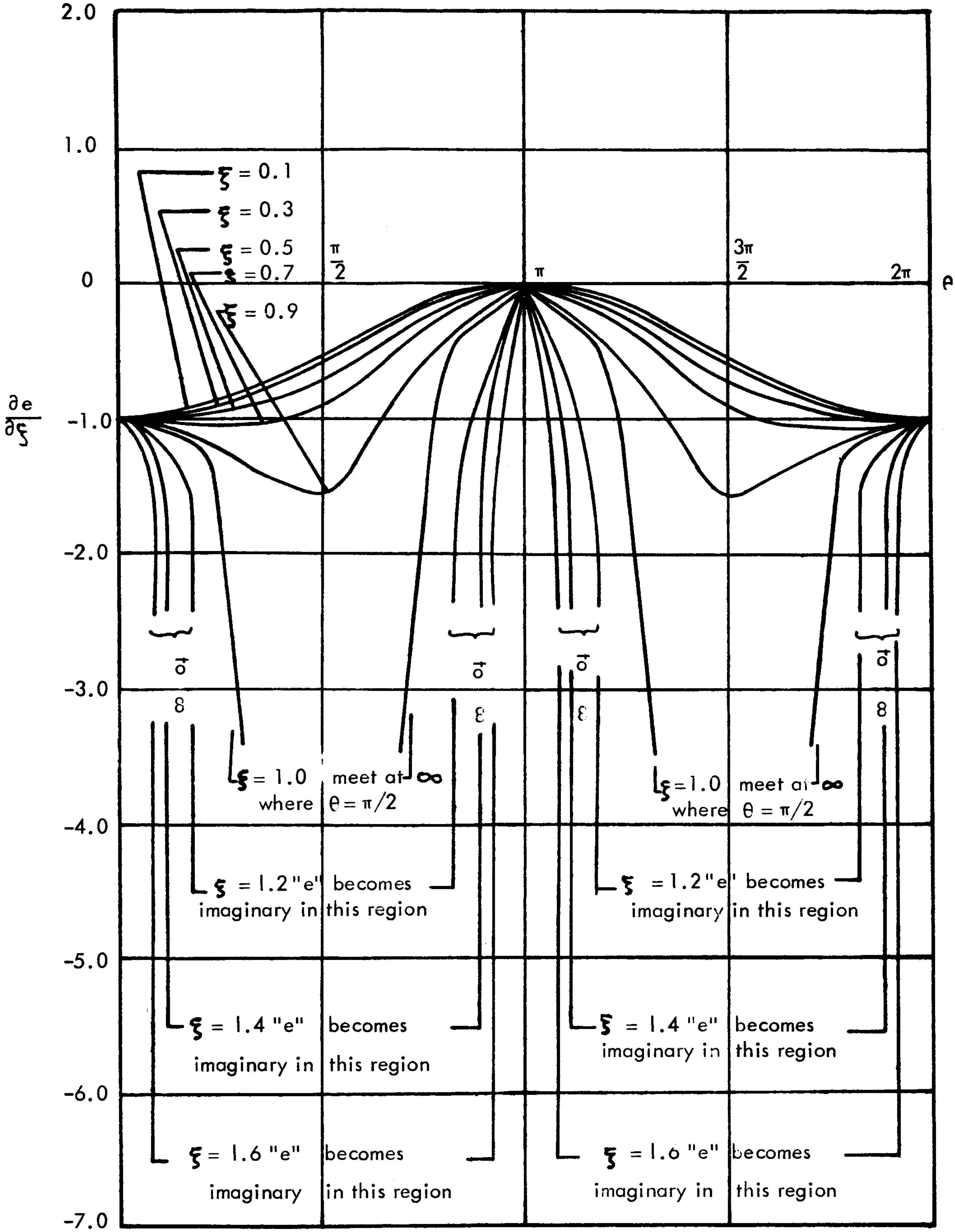


Figure 3 Values of $\frac{\partial e}{\partial \xi}$ at which $\xi =$ const. as a function of θ.

II. OPTIMIZATION OF INDIVIDUAL PARAMETERS

In the preceding work (part 2 No. 2 and 1, #1) the determining equations for errors in the orbital parameters due to errors in certain variables have been presented from essentially two approaches. The relationships are, in many cases, quite complicated. In any attempt to optimize a given set of orbital parameters, one would like to know the behavior of the errors of the parameters as a function of the determining variables. That is, when an error in a given parameter is minimum, and when it is maximum under a given set of measurements. This analysis may be accomplished by either of two methods. (1) The equation for an error may be differentiated partially and set equal to zero, and thus determine under what conditions an extremum of the given error exists. (2) The equations for all the parameters may be programmed on a computer and the minimum errors determined by varying the values of the independent variables over some range of values. Each of these methods has its advantages and disadvantages. We describe below how both techniques are being employed.

A. Minimization of Errors by Differentiation

The conditions have been determined under which errors occuring in the orbital parameters are minimum as determined by three position measurments. These results were obtained by differentiation of the equations for the various errors and setting the expression equal to zero, thus determining the conditions under which an extremum exists. The results of this calculation are given below, and details of the analysis for one parameter are shown.

Let us first examine the error of $\frac{\partial P}{\partial r_1}$ from Report No. 2, Page 15

$$\frac{\partial p}{\partial r_1} = - r_2^2 r_3^2 \sin \Delta \theta_{23} \left[\sin \left(\frac{\Delta \theta_{12}}{2}\right) \sin \left(\frac{\Delta \theta_{23}}{2}\right) \sin \left(\frac{\Delta \theta_{31}}{2}\right) \right] / [\delta^2] \qquad (48)$$

where

$$\Delta \theta_{12} = \theta_2 - \theta_1 , \quad \Delta \theta_{23} = \theta_3 - \theta_2 , \quad \Delta \theta_{31} = \theta_1 - \theta_3$$

and

$$\delta = \frac{r_1 r_2}{2} \sin(\theta_2 - \theta_1) + \frac{r_2 r_3}{2} \sin(\theta_3 - \theta_2) + \frac{r_3 r_1}{2} \sin(\theta_1 - \theta_3) .$$

Let

$$\frac{\partial p}{\partial r_1} = f(r_1, r_2, r_3, \theta_1, \theta_2, \theta_3) .$$

The necessary conditions for having an extreme value of $\frac{\partial p}{\partial r_1}$ will be

$$\frac{\partial f}{\partial r_1} = \frac{\partial f}{\partial r_2} = \frac{\partial f}{\partial r_3} = \frac{\partial f}{\partial \theta_1} = \frac{\partial f}{\partial \theta_2} = \frac{\partial f}{\partial \theta_3} = 0 .$$

Proceeding with these partial derivatives, we will obtain six equations such as

$$\frac{\partial f}{\partial r_1} = \{ -r_2^2 r_3^2 \sin \Delta\theta_{23} [\sin(\frac{\Delta\theta_{12}}{2}) \sin(\frac{\Delta\theta_{23}}{2}) \sin(\frac{\Delta\theta_{31}}{2})]\}$$

$$\{ -\frac{2\delta}{\delta 4} \frac{\partial \delta}{\partial r_1} \} = 0 . \quad (49)$$

$$\frac{r_2}{2} \sin(\theta_2 - \theta_1) + \frac{r_3}{2} \sin(\theta_1 - \theta_3) = 0 \quad (50)$$

$$\frac{\partial f}{\partial r_2} = \{ -2 r_2 r_3^2 \sin(\Delta\theta_{23}) [\sin(\frac{\Delta\theta_{12}}{2}) \sin(\frac{\Delta\theta_{23}}{2}) \sin(\frac{\Delta\theta_{31}}{2})]\} \frac{1}{\delta^2}$$

$$- \{ -r_2^2 r_3^2 \sin(\Delta\theta_{23}) [\sin(\frac{\Delta\theta_{12}}{2}) \sin(\frac{\Delta\theta_{23}}{2}) \sin(\frac{\Delta\theta_{31}}{2})]\} \frac{2\delta}{\delta^4} \frac{\partial \delta}{\partial r_2}$$

$$= \{ - r_2^{\,2}\, r_3^{\,2} \sin \Delta\theta_{23} \; [\, \sin (\frac{\Delta\theta_{12}}{2}) \sin (\frac{\Delta\theta_{23}}{2}) \sin (\frac{\Delta\theta_{31}}{2}) \,] \} \; \frac{2}{r_2 \,\delta^2}$$

$$-$$

$$- \{ - r_2^{\,2}\, r_3^{\,2} \sin \Delta\theta_{23} \; [\, \sin (\frac{\Delta\theta_{12}}{2}) \sin (\frac{\Delta\theta_{23}}{2}) \sin (\frac{\Delta\theta_{31}}{2}) \,] \} \; \frac{2}{\delta^3} \; \frac{\partial \delta}{\partial r_2} = 0$$

which gives

$$\frac{1}{r_2}\,\delta - \frac{\partial \delta}{\partial r_2} = 0 \tag{51}$$

$$\frac{r_1}{2} \sin (\theta_2 - \theta_1) + \frac{r_3}{2} \sin (\theta_3 - \theta_2) + \frac{r_3\, r_1}{2 r_2} \sin (\theta_1 - \theta_3)$$

$$- \frac{r_3}{2} \sin (\theta_3 - \theta_2) - \frac{r_1}{2} \sin (\theta_2 - \theta_1) = 0 \ .$$

Since

r_1 , r_2 , r_3 are not zero

then

$$\sin (\theta_1 - \theta_3) = 0 \tag{52}$$

Similarly, $\frac{\partial f}{\partial r_3} = 0 \qquad \frac{1}{r_3}\,\delta - \frac{\partial \delta}{\partial r_3} = 0$

$$\sin (\theta_2 - \theta_1) = 0 \ . \tag{53}$$

For

$$\frac{\partial f}{\partial \theta_1} = 0$$

$$\frac{\partial f}{\partial \theta_1} = \{ - r_2^2 r_3^2 \sin(\theta_3 - \theta_2) [\cos(\frac{\theta_2 - \theta_1}{2}) (-\frac{1}{2}) \sin(\frac{\theta_3 - \theta_2}{2}) \sin(\frac{\theta_1 - \theta_3}{2})$$

$$+ \sin(\frac{\theta_2 - \theta_1}{2}) \sin(\frac{\theta_3 - \theta_2}{2}) \cos(\frac{\theta_1 - \theta_3}{2}) \frac{1}{2}] \} \frac{1}{\delta^2}$$

$$+ \{ - 2 r_2^2 r_3^2 \sin \Delta\theta_{23} [\sin(\frac{\Delta\theta_{12}}{2}) \sin(\frac{\Delta\theta_{23}}{2}) \sin(\frac{\theta_{31}}{2})] \} \frac{2\delta}{\delta^4} \frac{\partial \delta}{\partial \theta_1}$$

$$= 0 \quad . \tag{54}$$

$$\frac{1}{2} [\cos(\frac{\theta_2 - \theta_1}{2}) \sin(\frac{\theta_3 - \theta_2}{2}) \sin(\frac{\theta_1 - \theta_3}{2}) - \cos(\frac{\theta_1 - \theta_3}{2}) \sin(\frac{\theta_2 - \theta_1}{2})$$

$$\sin(\frac{\theta_3 - \theta_2}{2})] \delta + [- \sin(\frac{\theta_2 - \theta_1}{2}) \sin(\frac{\theta_3 - \theta_2}{2}) \sin(\frac{\theta_1 - \theta_3}{2})] [- r_1 r_2 \cos(\theta_2 - \theta_1)$$

$$+ r_3 r_1 \cos(\theta_1 - \theta_3)] = 0. \tag{55}$$

$$\frac{1}{4} r_1 r_2 \cos(\frac{\theta_2 - \theta_1}{2}) \sin(\frac{\theta_1 - \theta_3}{2}) \sin(\theta_2 - \theta_1) + \frac{r_2 r_3}{4} \cos(\frac{\theta_2 - \theta_1}{2}) +$$

$$\sin(\frac{\theta_1 - \theta_3}{2}) \sin(\theta_3 - \theta_2) + \frac{r_3 r_1}{4} \cos(\frac{\theta_2 - \theta_1}{2}) \sin(\frac{\theta_1 - \theta_3}{2}) \sin(\theta_1 - \theta_3)$$

$$- \frac{r_1 r_2}{4} \cos(\frac{\theta_1 - \theta_3}{2}) \sin(\frac{\theta_2 - \theta_1}{2}) \sin(\theta_2 - \theta_1)$$

$$- \frac{r_2 r_3}{4} \cos\left(\frac{\theta_1 - \theta_3}{2}\right) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) (\theta_3 - \theta_2)$$

$$- \frac{r_3 r_1}{4} \cos\left(\frac{\theta_1 - \theta_3}{2}\right) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin(\theta_1 - \theta_3)$$

$$+ r_1 r_2 \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right) \cos(\theta_2 - \theta_1)$$

$$- r_3 r_1 \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right) \cos(\theta_1 - \theta_3) = 0 \ . \qquad (56)$$

$$\frac{\partial f}{\partial \theta_2} = \{ + r_2^2 r_3^2 \cos(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)$$

$$- r_2^2 r_3^2 \sin(\theta_3 - \theta_2) \cos\left(\frac{\theta_2 - \theta_1}{2}\right) \frac{1}{2} \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)$$

$$+ r_2^2 r_3^2 \sin(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \cos\left(\frac{\theta_3 - \theta_2}{2}\right) \frac{1}{2} \sin\left(\frac{\theta_1 - \theta_3}{2}\right) \} \frac{1}{\delta^2}$$

$$+ \{ -r_2^2 r_3^2 \sin\left(\frac{\theta_3 - \theta_2}{2}\right) [\sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)] \}$$

$$\frac{2\delta}{\delta^4} \frac{\partial \delta}{\partial \theta_2} = 0 \ . \qquad (57)$$

$$\{ r_2^2 r_3^2 \cos(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)$$

$$- \frac{r_2^2 r_3^2}{2} \sin(\theta_3 - \theta_2) \cos\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)$$

$$+ \frac{r_2^2 r_3^2}{2} \sin(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \cos\left(\frac{\theta_3 - \theta_1}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)\}$$

$$[\frac{r_1 r_2}{2} \sin(\theta_2 - \theta_1) + \frac{r_2 r_3}{2} \sin(\theta_3 - \theta_2) + \frac{r_3 r_1}{2} \sin(\theta_1 - \theta_3)]$$

$$[+ \frac{r_1 r_2}{2} \cos(\theta_2 - \theta_1) - \frac{r_2 r_3}{2} \cos(\theta_3 - \theta_2) = 0 \qquad (58)$$

$$\frac{\partial f}{\partial \theta_3} = \{ - r_2^2 r_3^2 \cos(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)$$

$$- r_2^2 r_3^2 \sin(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \frac{1}{2} \cos\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)$$

$$+ r_2^2 r_3^2 \sin(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \frac{-1}{2} \cos\left(\frac{\theta_1 - \theta_3}{2}\right)\} \frac{1}{\delta^2}$$

$$+\{ -r_2^2 r_3^2 \sin \Delta\theta_{23} [\sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)]\} \frac{2\delta}{\delta^4} \frac{\partial \delta}{\partial \theta_3} = 0$$

$$\{ - r_2^2 r_3^2 \cos(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)$$

$$- \frac{1}{2} r_2^2 r_3^2 \sin(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \cos\left(\frac{\theta_3 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)$$

$$+ \frac{r_2^2 r_3^2}{2} \sin(\theta_3 - \theta_2) \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_3}{2}\right) \} \delta$$

$$+ \left[\frac{r_2 r_3}{2} \cos(\theta_3 - \theta_2) - \frac{r_3 r_1}{2} \cos(\theta_1 - \theta_3) \right] = 0 \qquad (59)$$

From Eq. (52) and (53) we obtain

$$\theta_2 - \theta_1 = 0, \text{ or } \pi \qquad (60)$$

$$\theta_1 - \theta_3 = 0, \text{ or } \pi \qquad (61)$$

$$\theta_2 - \theta_3 = 2\pi \qquad (62)$$

using (60), (61),&(62)together with (56) yields

$$r_2 = r_3 \quad .$$

By the same token, from

$$r_1 = - r_3$$

also from (59) we see

$$r_1 = - r_2 \ .$$

From these results where

$$| r_1 | = | r_2 | = | r_3 |$$

$$\theta_1 - \theta_3 = \pi$$

$$\theta_2 - \theta_1 = \pi$$

These are quite clear to us. It will give us a circular orbit. Therefore, we conclude that the circular orbit will give us minimum error of $\left(\frac{\partial p}{\partial r_1}\right)$ from the measurements.

By the same token, we can demonstrate for other error quantities such as $\frac{\partial p}{\partial \theta_1}$, $\frac{\partial e}{\partial r_1}$, ... It will give us the same conclusion since the method is quite straight forward. We are going to omit the performance here.

B. Minimization Study By Numerical Evaluation

Due to the complexity of the preceding equations, the optimization of a given parameter results in a very lengthy calculation, as can be seen by results in Part A above. However, the behavior of the equations may easily be investigated by numerical techniques. For this approach, the equations for the determination of the orbital parameters by a particular set of measurements are programmed for a computer. With such a program, the effects of instrumental errors on the determination of the orbital parameter may easily be investigated by varying the observed quantities over some range which corresponds to the size of an observatorial error. This method has the added advantage of giving an exact number for the percent error in the determination of any given parameter for a given percentage error in the measured quantities. With this information, the values of the measured quantities may be varied over the entire range of the actual orbit and the conditions under which the error in a given parameter are minimum may be ascertained.

The numerical technique is being employed for the error minimization for both the position and dynamic variable methods of determining the parameters of an orbit. Results obtained to date are only of a preliminary nature, the object being chiefly that of checking the Fortran program on sample data. We will defer until later a description of the numerical minimization program and its application to the optimization of the orbital parameters.

III. REVIEW OF COORDINATE TRANSFORMATIONS

A. Consideration of A Station-Centered Coordinate System

All of the dynamic error analysis thus far presented has been in terms of variables and parameters in the earth centered coordinate system. However, actual measurements are made by tracking stations, or in relation to stations on the surface of the earth. Since the measurement of tracking error are known for the tracking station on the surface of the earth, the results obtained heretofore have to be modified, and the errors expressed in terms of the measurements in the station-centered coordinate system. Further, the location of this station with respect to the center of the earth is not exactly known. Thus, even if it were possible to obtain exact measurements at the tracking station, errors would be introduced in transforming to the dynamic, earth centered, coordinate system.

It is the purpose of this section to study the exact transformation procedure. In a future section, the error effects will be studied.

Consider a terrestrial station-centered or local coordinate system located on the surface of the earth at $\vec{R}_o$ from the center of the earth. Let $\vec{R}_o$ rotate with angular velocity $\vec{\Omega}$, let $\vec{R}'$ denote a point expressed in the local coordinate system, and let $\vec{R}$ denote the same point expressed in the fixed, earth centered, coordinate system. Let the fixed system be defined by the unit triad, $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, and the local, rotating system by $\hat{e}'_1$, $\hat{e}'_2$, $\hat{e}'_3$. These definitions are shown in Figure (4) .

Now, since $\vec{R}_o$, $\hat{e}'_i$ are fixed in magnitude, we have

$$\frac{d\vec{R}_o}{dt} = \vec{\Omega} \times \vec{R}_o , \quad \frac{d\hat{e}'_i}{dt} = \vec{\Omega} \times \hat{e}'_i \tag{63}$$

Let us have the following decompositions

$$\vec{R}' = R'_i \, \hat{e}'_i \tag{64}$$

$$\vec{R}_o = C_i(t)\,\hat{e}_i \tag{64}$$

$$\hat{e}'_i = a_{ij}(t)\,\hat{e}_j$$

where summation over repeated indices is used throughout.

The coordinates or the tracking station, are C_i (t) and a_{ij} (t) represents elements of the transformation matrix. We have

$$\vec{R} = \vec{R}_o + \vec{R}' = C_i\,\hat{e}_i + R'_i\,\hat{e}'_i \tag{65}$$

and find the geocentrical $\vec{R}$ expressed in terms of the local $\vec{R}$,

$$\vec{R} = C_j\,\hat{e}_j + R'_i\,a_{ij}\,\hat{e}_j = (C_j + R'_i\,a_{ij})\,\hat{e}_j \tag{66}$$

It is well known that

$$\vec{A} \times \vec{B} = e_{ijk}\,A_j\,B_k\,\hat{e}_i$$

where

$$e_{ijk} = \begin{cases} +\ 1\ ,\ ijk\ \text{ even permutation of 1, 2, 3} \\ -\ 1\ ,\ ijk\ \text{ odd permutation of 1, 2, 3} \\ 0\ \text{ otherwise} \end{cases}$$

and
$$\vec{A} = A_i\,\hat{e}_i\ ,\ \vec{B} = B_i\,\hat{e}_i$$

Thus we obtain, taking into account the rotation of the primed (station-centered) coordinate system

$$\frac{d\vec{R}}{dt} = \frac{dR_o}{dt} + \frac{dR'}{dt} = \Omega \times R_o + \Omega \times R' + \frac{dR'_i}{dt}\,\hat{e}'_i \tag{67}$$

$$= \vec{\Omega} \times \vec{R} + \frac{dR'_i}{dt} \hat{e}'_i \qquad (67)$$

$$= e_{ijk} \quad \Omega_j \, (C_k + R'_p \, a_{pk}) \; \hat{e}_i$$

$$+ \frac{dR'_p}{dt} \quad a_{pi} \; \hat{e}_i$$

The subscript indicates the component in the station-centered system. We find, in terms of the position in this system

$$\frac{d\vec{R}}{dt} = \left[\frac{dR'_p}{dt} \; a_{pi} + e_{ijk} \quad \Omega_j \; (C_k + R'_p \; a_{pr}) \right] \hat{e}_i \qquad (68)$$

Since we are concerned with the earth's rotation, we make e_3 the north pole and we write

$$\vec{\Omega} = \Omega \, \hat{e}_3 \qquad (69)$$

$$\Omega = 7.29 \times 10^{-5} \text{ rad/sec} \quad .$$

Thus

$$\vec{R} = (R'_i \; a_{ij} + C_j) \; \hat{e}_j \qquad (70)$$

$$\frac{d\vec{R}}{dt} = \left[e_{j3k} \; (R'_p \; a_{pk} + C_k) \; \Omega + \frac{dR'_p}{dt} \; a_{pj} \right] e_j \qquad (71)$$

In equations (70) and (71), a_{kj} and C_j depend on time. Let us determine this time dependence.

Since

$$\frac{d\vec{R}_o}{dt} = \vec{\Omega} \times \vec{R}_o \quad , \qquad (72)$$

or

$$\frac{d}{dt}\left[C_j(t)\ \hat{e}_j \right] = e_{ijk}\ \Omega_j\ C_k\ \hat{e}_i =$$

$$= e_{j3k}\ \Omega\ C_k\ \hat{e}_j = \frac{d\,C_j}{dt}\ \hat{e}_j\ , \tag{73}$$

we have

$$\frac{d\,C_j}{dt} = e_{j3k}\ C_k\ \Omega\ , \tag{74}$$

so that

$$\frac{dC_1}{dt} = -\,\Omega\, C_2\ ,$$

$$\frac{dC_2}{dt} = \Omega C_1\ , \tag{75}$$

$$\frac{dC_3}{dt} = 0\ .$$

Solving this system, we obtain

$$C_1(t) = C \cos(\Omega t + \delta)\ ,$$

$$C_2(t) = C \sin(\Omega t + \delta)\ , \tag{76}$$

$$C_3(t) = C_3 = \text{constant}\ ,$$

where C is a constant given by $C = (C_1^2 + C_2^2)^{1/2}$.

Similarly,

$$a_{i\,1}(t) = A_i \cos(\Omega t + \delta_i) \quad ,$$

$$a_{i\,2}(t) = A_i \sin(\Omega t + \delta_i) \quad , \tag{77}$$

$$a_{i\,3}(t) = a_{i\,3} = \text{constant} \ .$$

and the constants A_i are

$$A_i = (a^2_{i\,1} + a^2_{i\,2})^{1/2} \quad .$$

These results can be further simplified if we assume that at some time t_o the direction of the axes, $\hat{e}_i$ and $\hat{e}'_i$ of the coordinate systems coincide. We write with δ_{ij} for the Kronecker Delta.

$$a_{ij}(t_o) = \delta_{ij} \quad ,$$

and have in matrix notation

$$(a_{ij}) = \begin{pmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \sin(\omega) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{78}$$

$$\omega = \Omega(t - t_o)$$

which is clearly the transformation matrix for a rotation in the earth centered, equatorial plane.

B. Consideration of a Vehicle Born, Inertial Coordinate System

Consider next an inertial coordinate system located in the vehicle. Let $\vec{R}_o$ denote a point on the surface of the earth, expressed in the earth centered coordinate system, and let $\vec{R}$ denote the same point expressed in the vehicle born coordinate system. Let this point rotate with angular velocity $\vec{\Omega}$. Let $\vec{R}$ denote the vehicle location with respect to the center of the earth. Let the earth centered system be defined by the unit triad, $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, and the vehicle born system by $\hat{e}'_1$, $\hat{e}'_2$, $\hat{e}'_3$. These definitions are shown in Figure (5). As in Section A, we have

$$\frac{d\vec{R}_o}{dt} = \vec{\Omega} \times \vec{R}_o , \tag{79}$$

and

$$\vec{R}' = R'_i \, \hat{e}'_i ,$$

$$\vec{R}_o = C_i(t) \, \hat{e}_i , \tag{80}$$

$$\hat{e}'_i = a_{ij} \, \hat{e}_i .$$

However, since the primed coordinate system is inertial, the components of the transformation matrix a_{ij} are constant and we have

$$\vec{R} = \vec{R}_o - \vec{R}' = C_i \, \hat{e}_i - R'_i \, \hat{e}'_i . \tag{81}$$

We find for the geocentrical $\vec{R}$ expressed in terms of the vehicle measured R'_i

$$\vec{R} = (C_i - R'_i \, a_{ij}) \, \hat{e}_i . \tag{82}$$

Similarly, as in Section A

$$\frac{d\vec{R}}{dt} = \frac{d\vec{R}_o}{dt} - \frac{d\vec{R}'}{dt} = \vec{\Omega} \times \vec{R}_o - \frac{dR_i}{dt} \hat{e}'_i , \qquad (83)$$

or in terms of the geocentrical coordinate system, and using $\vec{\Omega} = \Omega \hat{e}_3$,

$$\frac{d\vec{R}}{dt} = \left[e_{i3k} \; \Omega \; C_k - \frac{dR'_p}{dt} \; a_{pi} \right] \hat{e}_i \; . \qquad (84)$$

The C_k (t) have the same form as given in Section A. If the directions or the axes $\hat{e}'_i$, $\hat{e}_i$ coincide, then we have the further simplification

$$a_{ij} = \delta_{ij}$$

where δ_{ij} is the kronecker delta, and obtain

$$\vec{R} = (C_i - R'_i) \; \hat{e}_i \; , \qquad (85)$$

and

$$\frac{d\vec{R}}{dt} = (e_{i\,3k} \; \Omega \; C_k - R'_i) \; \hat{e}_i \; . \qquad (86)$$

C. Consideration of A Vehicle Born, Orbit Aligned Coordinate System

The geometry of this system is the same as in Section II. Here, however, the primed coordinate system is taken to be the intrinsic or Frenét triad. In this system, $\hat{e}'_1$ is aligned with the tangent to the trajectory, $\hat{e}'_2$ is aligned with the inward normal, and $\hat{e}'_3$ is perpendicular to $\hat{e}'_1$ and $\hat{e}'_2$ in the right hand sense, i.e. $\hat{e}'_3 = \hat{e}'_1 \times \hat{e}'_2$. This is shown in Figure (6). The results are the same as in Section II, i.e.

$$R = (C_i - a_{ij} R'_i)\, \hat{e}_i \tag{87}$$

$$\frac{d\vec{R}}{dt} = \left[e_{i3k}\, \Omega C_k - \frac{dR'_p}{dt}\, a_{pi} \right] \hat{e}_i \ . \tag{88}$$

Here the transformation matrix a_{ij} is a function of time. Let us determine this dependence.

The geometry of the two body problem in three dimensions is shown in Figure (7). The intersection of the orbital plane with the equatorial plane is called the line of nodes. Ω is the angle between the line of nodes and the $\hat{e}_1$ axis in the equatorial plane. i is the inclination of the orbital plane with respect to the equatorial plane. θ_o is the angle between the perigee point and the line of nodes in the orbital plane. The orbit is given by

$$R = \frac{P}{1 + e \cos(\theta - \theta_o)} \ ,$$

where θ is measured in the direction of motion from the line of nodes in the orbital plane.

Now $\hat{e}'_3$ is perpendicular to the orbital plane determined by Ω and i. Clearly from Figure (7)

$$\hat{e}'_3 = \sin(i)\sin(\Omega)\,\hat{e}_1 - \sin(i)\cos(\Omega)\,\hat{e}_2 + \cos(i)\,\hat{e}_3 \quad , \tag{89}$$

and consequently, with $a_{ij} = \hat{e}'_i \cdot \hat{e}_j$

$$\begin{aligned} a_{31} &= \sin(i)\sin(\Omega) \\ a_{32} &= -\sin(i)\cos(\Omega) \\ a_{33} &= \cos(i) \quad . \end{aligned} \tag{90}$$

In the orbital plane consider a polar coordinate system, $\hat{e}_r$, $\hat{e}_\theta$ as shown in Figure (8), we have

$$\vec{R} = R\,\hat{e}_r \quad . \tag{91}$$

The tangent vector $\hat{e}'_1$ is given by

$$\hat{e}'_1 = \frac{d\vec{R}/d\theta}{\left| d\vec{R}/d\theta \right|} \quad . \tag{92}$$

Now

$$\frac{d\vec{R}}{d\theta} = \frac{dR}{d\theta}\,\hat{e}_r + R\,\frac{d\hat{e}_r}{d\theta} \quad , \tag{93}$$

with

$$\frac{dR}{d\theta} = \frac{Pe \sin(\theta - \theta_o)}{(1 + e \cos(\theta - \theta_o))^2} = \frac{R^2 e}{P} \sin(\theta - \theta_o) , \tag{94}$$

and

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta . \tag{95}$$

Thus

$$\frac{d\vec{R}}{d\theta} = \frac{R^2 e}{P} \sin(\theta - \theta_o)\ \hat{e}_r + R\ \hat{e}_\theta , \tag{96}$$

and

$$\left|\frac{d\vec{R}}{d\theta}\right| = \left[\left(\frac{R^2 e}{P} \sin(\theta - \theta_o)\right)^2 + R^2\right]^{1/2}$$

$$= R\left[\frac{R^2 e^2}{P^2} \sin^2(\theta - \theta_o) + 1\right]^{1/2} \tag{97}$$

$$= \frac{R^2}{1 + e \cos(\theta - \theta_o)} \left[2(1 + e \cos(\theta - \theta_o)) - (1 - e^2)\right]^{1/2}$$

$$= \frac{R^2}{P}\left[\frac{2}{R} - \frac{1}{a}\right]^{1/2} = \frac{R^2 U}{P K} = \frac{R^2 U}{C} .$$

Thus

$$\hat{e}_1' = \frac{Ce}{pu} \sin(\theta - \theta_o)\ \hat{e}_r + \frac{C}{uR}\ \hat{e}_\theta . \tag{98}$$

We have the following decompositions for e_r and e_θ

$$\hat{e}_r = \left[\cos(\theta) \cos(\Omega) - \sin(\theta) \cos(i) \sin(\Omega)\right] \hat{e}_1 \tag{99}$$

$$+ \left[\cos(\theta)\sin(\Omega) + \sin(\theta)\cos(i)\cos(\Omega)\right]\hat{e}_2 \tag{99}$$

$$+ \sin(\theta)\sin(i)\hat{e}_3$$

$$\hat{e}_\theta = -\left[\sin(\theta)\cos(\Omega) + \cos(\theta)\cos(i)\sin(\Omega)\right]\hat{e}_1$$

$$+ \left[-\sin(\theta)\sin(\Omega) + \cos(\theta)\cos(i)\cos(\Omega)\right]\hat{e}_2 \tag{100}$$

$$+ \cos(\theta)\sin(i)\hat{e}_3 \quad .$$

In these equations $\theta = \theta(t)$, which can be obtained from Keplers equation. We then have for the a_{ij}

$$a_{11} = \frac{Ce}{Pu}\sin(\theta - \theta_o)\left[\cos(\theta)\cos(\Omega) - \sin(\theta)\cos(i)\sin(\Omega)\right]$$

$$- \frac{C}{uR}\left[\sin(\theta)\cos(\Omega) + \cos(\theta)\cos(i)\sin(\Omega)\right]$$

$$a_{12} = \frac{Ce}{Pu}\sin(\theta - \theta_o)\left[\cos(\theta)\sin(\Omega) + \sin(\theta)\cos(i)\cos(\Omega)\right] \tag{101}$$

$$+ \frac{C}{uR}\left[-\sin(\theta)\sin(\Omega) + \cos(\theta)\cos(i)\cos(\Omega)\right]$$

$$a_{13} = \frac{Ce}{Pu}\sin(\theta - \theta_o)\sin(\theta)\sin(i) + \frac{C}{uR}\cos(\theta)\sin(i) \quad .$$

$\hat{e}'_2$ may now be determined from the equation

$$\hat{e}'_2 = \hat{e}'_3 \times \hat{e}'_1 \tag{102}$$

or in transformation form

$$
\begin{aligned}
a_{21} &= a_{13}\ a_{32} - a_{12}\ a_{33} \\
a_{22} &= a_{11}\ a_{33} - a_{13}\ a_{31} \\
a_{23} &= a_{31}\ a_{12} - a_{32}\ a_{11}
\end{aligned}
\tag{103}
$$

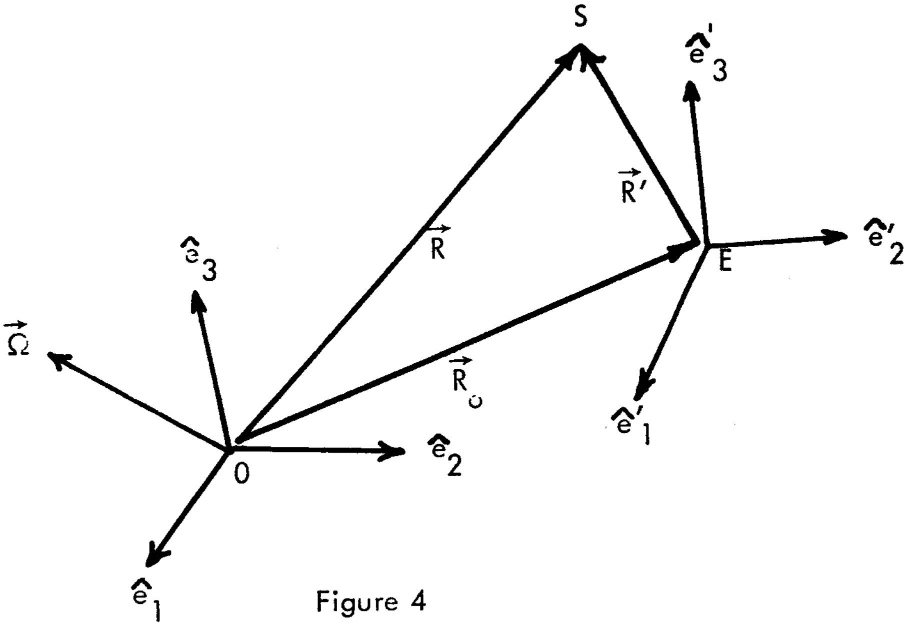


Figure 4

Geometry For Earth Fixed Tracking Station

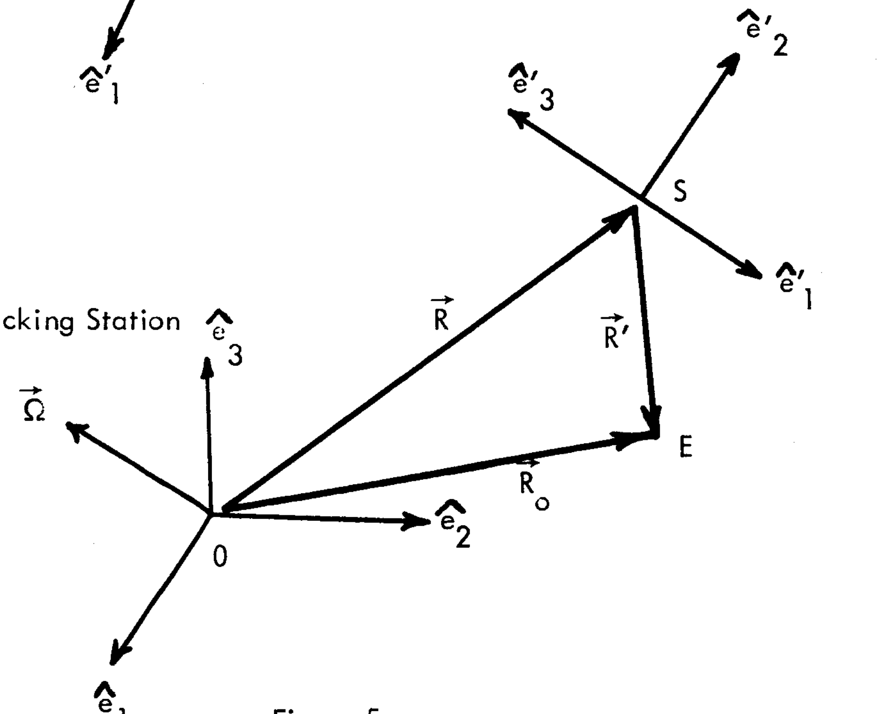
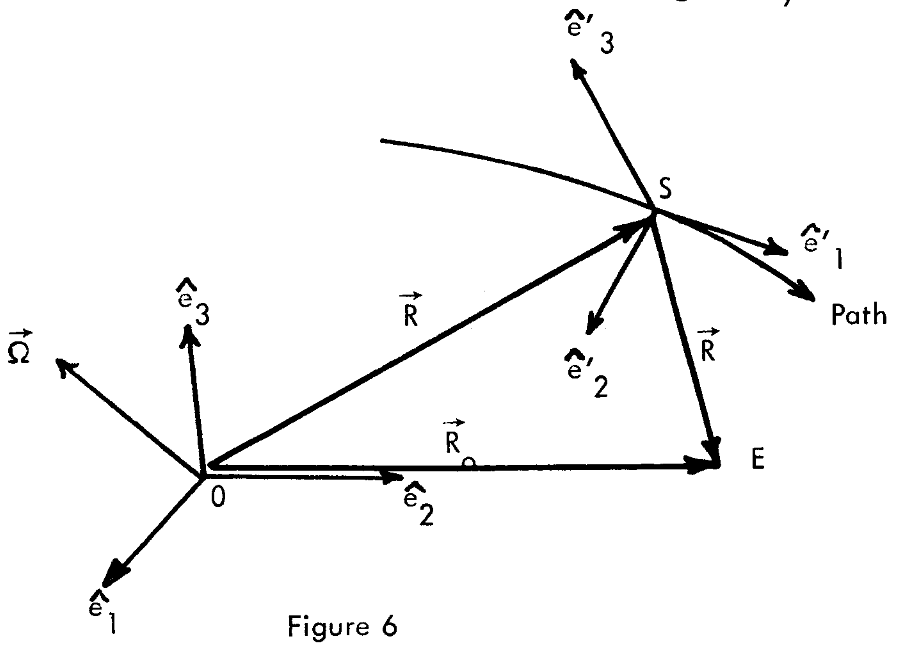


Figure 5

Geometry For Inertial Satellite Fixed Tracking Station



0 - Center of Earth

S - Satellite Position

E - Surface of Earth

Figure 6

Geometry For Intrinsic On Frenet Satellite Fixed Tracking Station

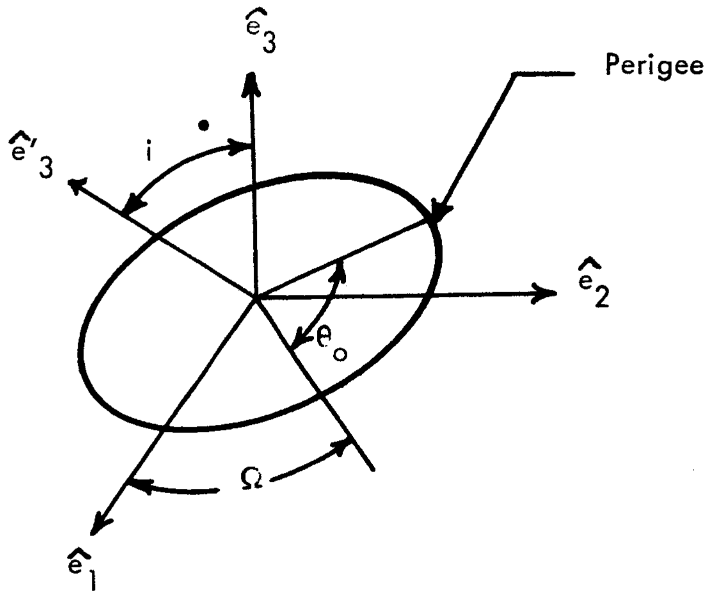


Figure 7

Orbit Geometry

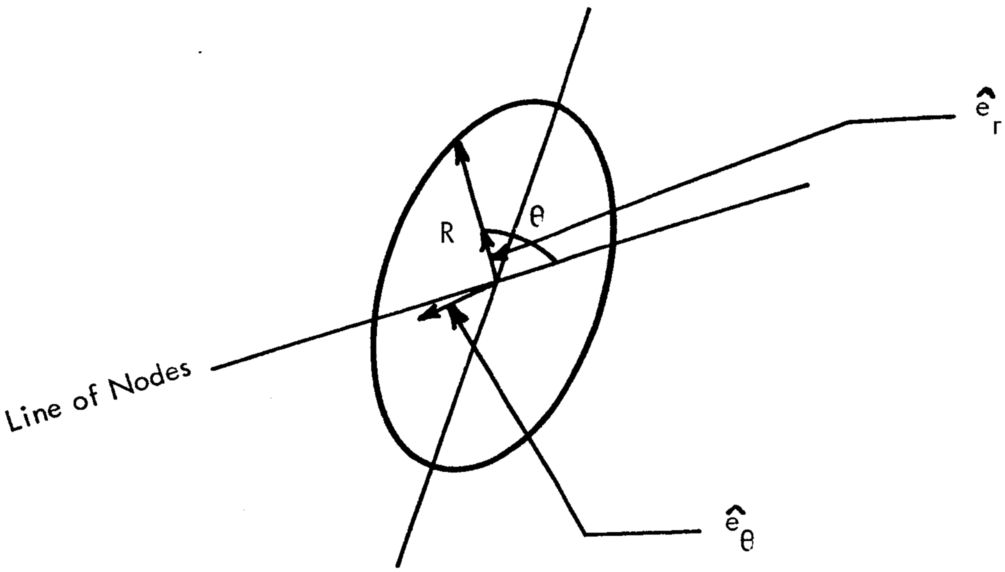


Figure 8

Orbital Plane Coordinate System

IV. APPENDIX

The following two tables serve the same purpose as Tables I and II in Report No. 2. They give a quick survey of the error analysis of the present report. The tabulation is the same as in the previous report with the exception of an additional column which contains the normalized parameter $\varsigma = \frac{r}{a}$ in the geometric case.

In this display, Columns 3 through 14 plus Column 16 contain the variables which are considered as measured quantities (observables) in the previous analysis (Report #1 and #2). In Column A is the error in one of the orbital parameters which is introduced by an error in one of the observables which error appears in Column B. The errors in the orbital parameters are functions of the measurement error (Column B) and of the other variables and parameters of the orbital equations which are indicated by an x in the appropriate Columns 3 through 14 and Column 16.

ERROR ANALYSIS

(Geometric Case)

(continued, see also Report No. 2, p. 11)

		Observables										Parameters					
		$\vec{r}_1$		$\vec{r}_2$		$\vec{r}_3$		$\vec{v}_1$		$\vec{v}_2$							
A	B	r_1	θ_1	r_2	θ_2	r_3	θ_3	$\dot{r}_1$	$\dot{\theta}_1$	$\dot{r}_2$	$\dot{\theta}_2$	a	e	θ_o	$\xi = \frac{r}{a}$	Eq. No	Remarks
Δe	Δr	×	×									×	×			1	$\theta_o = 0$
Δe	$\Delta \xi$		×										×		×	37	$\theta_o = 0$
Δe	$\Delta \theta$	×	×									×	×			5	$\theta_o = 0$
			×												×	40	$\theta_o = 0$
Δp	Δr	×										×	×			1a	$\theta_o = 0$
			×										×			2	$\theta_o = 0$
Δp	$\Delta \theta$	×	×									×	×			6	$\theta_o = 0$
Δp	$\Delta \xi$		×									×	×		×	39	$\theta_o = 0$
Δa	Δr	×	×									×	×			3	$\theta_o = 0$
$\Delta \theta_o$	Δr	×	×									×	×			4	$\theta_o \neq 0$

TABLE I

ERROR ANALYSIS

(Dynamic Case)

(continued, see also Report No. 2, p. 11)

A	B	$\vec{r}_1$		$\vec{r}_2$		$\vec{r}_3$		$\vec{v}_1$		$\vec{v}_2$						
		Observables										Parameters			Eq. No.	Remarks
		r_1	θ_1	r_2	θ_2	r_3	θ_3	$\dot{r}_1$	$\dot{\theta}_1$	$\dot{r}_2$	$\dot{\theta}_2$	a	e	ρ_o		
Δp	Δr_1	x	x					x		x					13	
Δp	$\Delta \dot{r}_1$	x		x				x							15	
Δe	Δr_1	x		x				x		x		x	x		17	
Δe	$\Delta \dot{r}_1$	x		x				x				x	x		18	
$\Delta \theta_o$	Δr_1	x	x	x	x			x		x		x	x		21	
$\Delta \theta_o$	$\Delta \dot{r}_1$	x	x	x	x			x				x	x		22	
$\Delta \theta_o$	$\Delta \rho_1$	x	x	x	x							x	x		24	
ΔP	Δr_1	x							x						28	
Δp	$\Delta \dot{\theta}_1$	x							x						29	
Δe	Δr_1	x	x						x					x	31	
Δe	$\Delta \dot{\theta}$	x	x						x						32	
Δe	$\Delta \theta_1$	x							x					x	33	

TABLE 2

V. ERRATA SHEET

for the Report No. 2 of "Parameter Optimization "

Page 7, last equation — Should read:

$$\dot{\theta}_2 = \frac{A}{r_2^2} \quad \frac{A}{(r_1 + \Delta r)_1^{\,2}}$$

$$\dot{\theta}_2 = \frac{A}{r_2^2} = \frac{A}{(r_1 + \Delta r_1)^2}$$

Page 21, 8th line

$$\frac{-R^2}{2} \Delta\theta_{12} \quad \Delta\theta_{23} \quad \Delta\theta_{31}$$

$$\frac{-R^2}{4} \Delta\theta_{12} \quad \Delta\theta_{23} \quad \Delta\theta_{31} .$$

Page 22, last equation

$$K a e \frac{\partial e}{\partial v} = a^2 v \left(1-e^2\right) - e \frac{\partial C}{\partial V}$$

$$K a e \frac{\partial e}{\partial v} = a^2 v \left(1-e^2\right) - C \frac{\partial C}{\partial V}$$

Page 23, second equation

$$K a V \frac{\partial e}{\partial V} = a^2 V^2 (1-e^2) - C^2$$

$$K a e V \frac{\partial e}{\partial V} = a^2 V^2 (1-e^2) - C^2$$

Page 24, second equation

$$2C = \frac{\partial C}{\partial r} = K (1 + e \cos \theta) + K \cos \theta \frac{\partial e}{\partial r}$$

$$2C \frac{\partial C}{\partial r} = K (1 + e \cos \theta) + K r \cos \theta \frac{\partial e}{\partial r}$$

Page 24, 7th equation

$$\frac{\partial p}{\partial r} = \frac{2e}{K} \quad \frac{\partial e}{\partial r} = \frac{4C^2}{K} = \frac{4p}{r}$$

$$\frac{\partial p}{\partial r} = \frac{2C}{K} \frac{\partial C}{\partial r} = \frac{4C^2}{K} = \frac{4p}{r}$$

Should read

Page 25, 5th equation

$$\frac{\partial p}{\partial r} = 1 + e\cos(\theta - \theta_o) + re\frac{\partial e}{\partial r}\cos(\theta - \theta_o) + er\sin(\theta - \theta_o)\frac{\partial \theta_o}{\partial r} = \frac{p}{r} + \frac{3C^2 e}{K} + er\sin(\theta - \theta_o)\frac{\partial \theta_o}{\partial r}$$

$$\frac{\partial p}{\partial r} = 1 + e\cos(\theta - \theta_o) + r\frac{\partial e}{\partial r}\cos(\theta - \theta_o) + er\sin(\theta - \theta_o)\frac{\partial \theta_o}{\partial r} = \frac{p}{r} + \frac{3C^2}{K} + er\sin(\theta - \theta_o)\frac{\partial \theta_o}{\partial r}$$

Page 25, 6th equation

$$\frac{3p}{r} - \frac{3C^2 e}{K} = er\sin(\theta - \theta_o)\frac{\partial \theta_o}{\partial r}$$

$$\frac{3p}{r} - \frac{3C^2}{K} = er\sin(\theta - \theta_o)\frac{\partial \theta_o}{\partial r}$$

Page 25, 7th equation

$$\frac{\partial \theta_o}{\partial r} = \frac{3}{er\sin(\theta - \theta_o)}\quad\frac{p}{r} - \frac{C^2 e}{K}$$

$$\frac{\partial \theta_o}{\partial r} = \frac{3}{er\sin(\theta - \theta_o)}\quad\frac{p}{r} - \frac{C^2}{K}$$

Page 27, 3rd equation

$$\frac{\partial p}{\partial \theta} = r\cos(\theta - \theta)\frac{\partial e}{\partial \theta}$$

$$\frac{\partial p}{\partial \theta} = r\cos(\theta - \theta_o)\frac{\partial e}{\partial \theta}$$